

# Matrix integrals as Borel sums of Schur function expansions <sup>†</sup>

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## Abstract

The partition function for unitary two matrix models is known to be a double KP  $\tau$ -function, as well as providing solutions to the 2-dimensional Toda hierarchy. It is shown how it may also be viewed as a Borel sum regularization of divergent sums over products of Schur functions in the two sequences of associated KP flow variables.

## 1. Introduction. $\tau$ -functions related to Schur functions.

We recall here a method for constructing a large class of double KP  $\tau$ -functions in terms of infinite sums over Schur functions, in which the two sets of flow variables appear on a symmetrical footing. (See [O, OS] for further details of such constructions, and additional applications.)

### 1a. Schur function expansion of $\tau$ functions.

Suppose we are given:

- (i) A sequence  $\{r(m)\}_{m \in \mathbf{N}}$  of complex numbers.
- (ii) A partition  $\lambda = (n_1 \geq \dots \geq n_p \geq 0)$  of  $n = \sum_{i=1}^p n_i$ .

Define, for each integer  $M$ , the quantity

$$r_\lambda(M) := \prod_{i,j} r(M + j - i), \quad \forall i, j \in \lambda, \quad (1.1)$$

where the product is over all pairs  $(i, j)$  that lie within a cell in the corresponding Young diagram (labelled like the entries of a matrix).

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Using the quantities  $r_\lambda(M)$  as coefficients, we may define an infinite sum over partitions (without, for the moment, considering the question of domain of convergence):

$$\tau_r(M, \mathbf{t}, \mathbf{t}^*) := \sum_{\lambda} r_\lambda(M) s_\lambda(\mathbf{t}) s_\lambda(\mathbf{t}^*), \quad (1.2)$$

where  $\mathbf{t} = (t_1, t_2, \dots)$  and  $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$  are two infinite sequences of flow variables, and the Schur function  $s_\lambda(\mathbf{t})$  is defined by

$$s_\lambda(\mathbf{t}) = \det(h_{n_i+j-i}(\mathbf{t})) \quad (1.3)$$

in terms of the elementary Schur functions (complete symmetric functions)  $\{h_j(\mathbf{t})\}_{j \in \mathbf{N}}$ , which are determined by:

$$\exp\left(\sum_{m=0}^{\infty} t_m z^m\right) = \sum_{m=1}^{\infty} z^m h_m(\mathbf{t}). \quad (1.4)$$

Whenever the sum (1.2) is convergent, it may be shown to be a double KP  $\tau$ -function [O, OS]. An indication of how this is done, using the fermionic Fock space construction, is given in the next subsection.

### 1b. Free fermion construction of the $\tau$ function.

Introduce, on a suitably defined fermionic Fock space, the free fermion creation and annihilation operators  $\{\psi_j, \psi_j^*\}_{j \in \mathbf{Z}}$ , which satisfy the usual anti-commutation relations

$$\begin{aligned} [\psi_j, \psi_k]_+ &= 0, & [\psi_j^*, \psi_k^*]_+ &= 0, \\ [\psi_j, \psi_k^*]_+ &= \delta_{jk}, & j, k &\in \mathbf{Z}. \end{aligned} \quad (1.5)$$

Then for any given sequence  $\{r(m)\}_{m \in \{bfN\}}$  we have the following fermionic formula for the associated  $\tau$ -function:

$$\tau_r(M, \mathbf{t}, \mathbf{t}^*) = \langle M | e^{H(\mathbf{t})} e^{-A(\mathbf{t}^*)} | M \rangle, \quad (1.6)$$

where

$$H(\mathbf{t}) := \sum_{n=1}^{\infty} t_n H_n, \quad A(\mathbf{t}^*) := \sum_{n=1}^{\infty} t_n^* A_n, \quad (1.7a)$$

$$H_n := \sum_{k=-\infty}^{\infty} \psi_k \psi_{k+n}^*, \quad n \neq 0, \quad (1.7b)$$

$$A_n := \sum_{k=-\infty}^{\infty} r(k) r(k-1) \dots r(k-n+1) \psi_{k-n}^* \psi_k, \quad (1.7c)$$

and  $|M\rangle$  is the charge  $M$  vacuum state defined by

$$\begin{aligned} |M\rangle &:= \psi_{M-1} \cdots \psi_1 \psi_0 |0\rangle, \quad M > 0, \\ |M\rangle &:= \psi_M^* \cdots \psi_{-2}^* \psi_{-1}^* |0\rangle, \quad M < 0. \end{aligned} \tag{1.8}$$

Here, the operator  $e^{-A(\mathbf{t}^*)}$  may be seen as preparing the initial state, for fixed values of the parameters  $\mathbf{t}^*$ , while the operators  $e^{-H(\mathbf{t})}$  generate the abelian group action determining the KP flows. (All the  $H_n$ 's commute amongst themselves, as do the  $A_n$ 's.) In this sense, the parameters  $\mathbf{t}^*$  are just viewed as distinguishing the elements of an infinite family of KP  $\tau$ -functions in the  $\mathbf{t}$  variables. However, the rôles of the two sets of parameters  $\mathbf{t}$  and  $\mathbf{t}^*$  may be interchanged and  $\tau_r(M, \mathbf{t}, \mathbf{t}^*)$  may also be interpreted as a KP  $\tau$ -function in the  $\mathbf{t}^*$  flow variables for fixed values of  $\mathbf{t}$ , thereby defining a double KP  $\tau$ -function. Moreover, the whole sequence of  $\tau_r(M, \mathbf{t}, \mathbf{t}^*)$ 's for successive values of  $M$  may be shown to define solutions of the 2-dimensional Toda hierarchy [O, OS].

In what follows, we restrict to the particular choice

$$r(n) = n, \tag{1.9}$$

which gives

$$r_\lambda(M) = (M)_\lambda := \prod_{j=1}^p (M - j)_{n_j}, \tag{1.10}$$

where

$$(N)_m := N(N+1) \cdots (N+m-1), \tag{1.11}$$

is the Pochhammer symbol, and denote the corresponding  $\tau$ -function

$$\tau_{\text{Id}}(M, \mathbf{t}, \mathbf{t}^*) =: \mathbf{F}_\lambda^M(\mathbf{t}, \mathbf{t}^*) = \sum_{\lambda} (M)_\lambda s_\lambda(\mathbf{t}) s_\lambda(\mathbf{t}^*). \tag{1.12}$$

It is easy to see that for most values of the parameters  $\mathbf{t}$  this is actually a divergent sum, and hence needs some interpretation. This will be given in the next section. For the present, we just note that there do exist finite dimensional subregions in the parameter space in which the sum converges; in fact, there are regions where only a finite number of terms do not vanish, and hence  $\mathbf{F}_\lambda^M(\mathbf{t}, \mathbf{t}^*)$  is actually a polynomial in each of the flow variables. Namely, if we choose one of the infinite sets of deformation parameters, say  $\mathbf{t}$ , to be given in terms a finite set of quantities  $\{x_1, \dots, x_P\}$  by:

$$t_j := -\frac{1}{j} \sum_{m=1}^P x_m^j, \tag{1.13}$$

the elementary Schur functions  $h_m(\mathbf{t})$  vanish identically for  $m > P$ . Therefore the sums (1.12) become finite, and  $\mathbf{F}_\lambda^M(\mathbf{t}, \mathbf{t}^*)$  is just a polynomial in the parameters  $(t_1, t_2, \dots)$  and  $(t_1^*, t_2^*, \dots)$ .

For  $M = 1$ , (1.12) takes a particularly simple form, since we then have  $r_{(n)}(1) = n!$ , and the only partition entering is  $\lambda = (n)$ , so (1.12) reduces to the expression

$$\mathbf{F}^1(\mathbf{t}, \mathbf{t}^*) = \sum_{n=0}^{\infty} n! h_n(\mathbf{t}) h_n(\mathbf{t}^*) . \quad (1.14)$$

**1c. Determinant expression for  $\mathbf{F}^M(\mathbf{t}, \mathbf{t}^*)$  in terms of  $\mathbf{F}^1(\mathbf{t}, \mathbf{t}^*)$  .**

It is possible to deduce a simple expression for  $\mathbf{F}^M(\mathbf{t}, \mathbf{t}^*)$  as a finite determinant in terms of  $\mathbf{F}^1(\mathbf{t}, \mathbf{t}^*)$  and its derivatives.

**Proposition 1.1.** *In any region where the sum (1.12) is uniformly convergent with respect to the parameters  $t_1$  and  $t_1^*$  the  $\tau$ -function  $\mathbf{F}^M(\mathbf{t}, \mathbf{t}^*)$  may be expressed in terms of  $\mathbf{F}^1(\mathbf{t}, \mathbf{t}^*)$  through the following formula:*

$$\mathbf{F}^M(\mathbf{t}, \mathbf{t}^*) = \frac{1}{\prod_{j=1}^{M-1} j!} \det \left( \frac{\partial^{a+b} \mathbf{F}^1(\mathbf{t}, \mathbf{t}^*)}{\partial t_1^a \partial t_1^{*b}} \right)_{\{a,b=0, \dots, M-1\}} . \quad (1.15)$$

This result, and another one similar to it, discussed in the next section, may be obtained as a consequence of the following well-known lemma:

**Lemma 1.2:** *For any measure  $d\mu(x, y)$  (real or complex) and any two sets of functions  $\{\phi_i(x), \psi_i(y)\}_{i=1, \dots, M}$ , if the the following integrals exist along suitably chosen contours  $\Gamma_x$  and  $\tilde{\Gamma}_y$  in the complex  $x$ - and  $y$ -planes, they satisfy the identity*

$$\begin{aligned} & \int_{\Gamma_x} \int_{\tilde{\Gamma}_y} d\mu(x_1, y_1) \dots \int_{\Gamma_x} \int_{\tilde{\Gamma}_y} d\mu(x_M, y_M) \det(\phi_i(x_j)) \det(\psi_k(y_l)) \\ &= M! \det \left( \int_{\Gamma_x} \int_{\tilde{\Gamma}_y} d\mu(x, y) \phi_i(x) \psi_j(y) \right) . \end{aligned} \quad (1.16)$$

The validity of eq. (1.15) may also be shown for more general parameter values, either as equalities between formal series or in the sense of asymptotic expansions about a Gaussian point ( $t_2 \neq 0$ ,  $t_j = 0$ ,  $\forall j > 2$ ).

**Proof of Proposition 1.1:** Use

$$\frac{\partial^a h_n}{\partial t_1^a} = h_{n-a}, \quad (1.17)$$

(where  $h_j$  is understood to vanish for negative  $j$ ) to deduce

$$\frac{\partial^{a+b} \mathbf{F}^1(\mathbf{t}, \mathbf{t}^*)}{\partial t_1^a \partial t_1^{*b}} = \sum_{m=1}^{\infty} m! h_{m-a}(\mathbf{t}) h_{m-b}(\mathbf{t}^*) , \quad (1.18)$$

and apply the discrete measure version of Lemma 1.2 with:

$$\phi_i(m) := h_{m-i+1}(\mathbf{t}), \quad \psi_j(m) := h_{m-j+1}(\mathbf{t}^*), \quad (1.19)$$

using (1.3).

## 2. Relation between $\mathbf{F}^M(\mathbf{t}, \mathbf{t}^*)$ and matrix integrals

### 2a. The 2-matrix partition function $\mathbf{Z}_{2P}^M$ .

The unitary 2-matrix model partition function for  $M \times M$  hermitian matrices [IZ, DKK, MS, BEH], after integrating out the “angular” variables, is given, up to a proportionality constant, by the following multiple integral over the eigenvalues.

$$\mathbf{Z}_{2P}^M(\mathbf{t}, \mathbf{t}^*) := \int_{\Gamma_{x_1} \times \Gamma_{y_1}} \cdots \int_{\Gamma_{x_M} \times \Gamma_{y_M}} \prod_{a=1}^M \Delta(\mathbf{x}) \Delta(\mathbf{y}) e^{\left(\sum_{j=1}^{\infty} (t_j x_a^j + t_j^* y_a^j - x_a y_a)\right)} dx_a dy_a, \quad (2.1)$$

where

$$\Delta(\mathbf{x}) := \prod_{i>j=1}^M (x_i - x_j), \quad \Delta(\mathbf{y}) := \prod_{i>j=1}^M (y_i - y_j).$$

(Although for the case of Hermitian matrices, the integration is along the real axes, we retain here the possibility of integration over more general contours  $\Gamma_x$  and  $\tilde{\Gamma}_y$  in the  $x$  and  $y$  planes.)

This may also be expressed as a determinant in terms of derivatives of  $\mathbf{Z}_{2P}^M(\mathbf{t}, \mathbf{t}^*)$ :

**Proposition 2.1:** *We have the following expression for the 2-matrix partition function in terms of the  $M = 1$  case, valid in any region where the integral (2.1) is uniformly convergent with respect to the parameters  $t_1$  and  $t_1^*$ .*

$$\mathbf{Z}_{2P}^M(\mathbf{t}, \mathbf{t}^*) = M! \det \left( \frac{\partial^{a+b} \mathbf{Z}_{2P}^1(\mathbf{t}, \mathbf{t}^*)}{\partial t_1^a \partial t_1^{*b}} \Big|_{a,b=0, \dots, M-1} \right). \quad (2.2)$$

The proof of this result is also a straightforward application of Lemma 1.2.

### 2b. Schur function expansion of $\mathbf{Z}_{2P}^M$ .

We now state the main result, relating the tau function  $\mathbf{F}^M(\mathbf{t}, \mathbf{t}^*)$  to the 2-matrix partition function  $\mathbf{Z}_{2P}^M(\mathbf{t}, \mathbf{t}^*)$ .

**Theorem 2.2:** *In any region where both the integral (2.1) defining the partition function  $\mathbf{Z}_{2P}^M(\mathbf{t}, \mathbf{t}^*)$  and the sum (1.12) defining the  $\tau$  function  $\mathbf{F}^M(\mathbf{t}, \mathbf{t}^*)$  are uniformly convergent, we have the equality:*

$$\mathbf{Z}_{2P}^M(\mathbf{t}, \mathbf{t}^*) = 2\pi \left( \prod_{j=1}^M j! \right) \mathbf{F}^M(\mathbf{t}, \mathbf{t}^*). \quad (2.3)$$

More generally, when the integral (2.1) defining  $\mathbf{Z}_{2P}^M(\mathbf{t}, \mathbf{t}^*)$  is convergent, whether the sum (1.12) is convergent or not, the former may be viewed as a multi-variable Borel sum of the series (1.12) defining  $\mathbf{F}^M(\mathbf{t}, \mathbf{t}^*)$ . Moreover, even when the integral is not convergent, it may be expanded in a perturbation series about a Gaussian point where  $t_2 \neq 0$ ,  $t_n = 0$ ,  $\forall n > 2$ , and the resulting asymptotic series is given by the sum (1.12). A proof of these result is given in [HO]. Here, we just give a sketch of how the identification as a Borel sum may be derived for the case  $M = 1$ , and conclude, from Props. 1.2 and 2.1 that this implies the analogous result in the multiple integral case. We also indicate a purely formal approach directly for arbitrary  $M$ , which does not address the question of convergence.

**2c. Formal expansions for  $M = 1$ .**

We first show how the relation (2.3) may be derived formally, without considerations of convergence, for the case  $M = 1$ .

**Proposition 2.3:** *Choosing the integration contours  $\Gamma_x$  and  $\tilde{\Gamma}_y$  to be orthogonal lines in the complex plane, the following identity holds:*

$$\mathbf{Z}_{2P}^1(\mathbf{t}, \mathbf{t}^*) = 2\pi i \mathbf{F}^1(\mathbf{t}, \mathbf{t}^*) . \quad (2.4)$$

as formal expansions in powers of the variables  $\mathbf{t} = (t_1, t_2, \dots)$  and  $\mathbf{t}^* = (t_1^*, t_2^*, \dots)$ .

**Proof:** Substituting the series (1.4) generating the elementary Schur functions into  $\mathbf{Z}_{2P}^1$  gives:

$$\begin{aligned} \mathbf{Z}_{2P}^1(\mathbf{t}, \mathbf{t}^*) &= \int_{\Gamma_x} dx \int_{\Gamma_y} dy \sum_{m=1}^{\infty} x^m h_m(\mathbf{t}) \exp\left(\sum_{j=1}^{\infty} t_j^* y^j\right) e^{-xy} \\ &= \int_{\Gamma_x} dx \int_{\Gamma_y} dy \sum_{m=1}^{\infty} h_m(\mathbf{t}) \exp\left(\sum_{j=1}^{\infty} t_j^* y^j\right) (-1)^m \frac{d^m}{dy^m} e^{-xy} \end{aligned} \quad (2.5)$$

Taking the derivatives with respect to  $y$  outside the  $x$  integral, and evaluating the  $x$  integral along the imaginary axis, we obtain a  $\delta$ -function, giving

$$\begin{aligned} \mathbf{Z}_{2P}^1(\mathbf{t}, \mathbf{t}^*) &= 2\pi i \int_{\Gamma_y} dy \sum_{m=1}^{\infty} (-1)^m h_m(\mathbf{t}) \frac{d^m}{dy^m} \delta(y) \exp\left(\sum_{j=1}^{\infty} t_j^* y^j\right) \\ &= 2\pi i \sum_{m=1}^{\infty} h_m(\mathbf{t}) \frac{d^m}{dy^m} \left( \sum_{j=1}^{\infty} h_j(\mathbf{t}^*) y^j \right)_{y=0} \\ &= 2\pi i \sum_{m=1}^{\infty} m! h_m(\mathbf{t}) h_m(\mathbf{t}^*) = 2\pi i \mathbf{F}^1(\mathbf{t}, \mathbf{t}^*) . \end{aligned} \quad (2.6)$$

**2d. Borel sum interpretation of eq. (2.3) for  $M = 1$ .**

We now show, for the case  $M = 1$ , how  $\mathbf{Z}_{2P}^1(\mathbf{t}, \mathbf{t}^*)$  may be interpreted as a Borel sum for (1.14) even when the sum diverges.

To do this, we reinterpret the integrals defining  $\mathbf{Z}_{2P}^1(\mathbf{t}, \mathbf{t}^*)$  in such a way that the variables  $x$  and  $y$  are replaced instead by a pair of complex conjugate variables  $z$  and  $\bar{z}$  in the complex plane, with the integral viewed simply as a double integral in the plane.

We then have, using the expansion (1.4):

$$\begin{aligned} \int \int dz d\bar{z} e^{\sum_{j=1}^{\infty} (t_j z^j t_j^* \bar{z}^j - z \bar{z})} &= \int_0^{\infty} r dr \int_0^{2\pi} d\phi \sum_{j=1}^{\infty} r^j h_j(\mathbf{t}) \sum_{k=1}^{\infty} r^k h_k(\mathbf{t}^*) e^{i(k-j)\phi} e^{-r^2} \\ &= 2\pi \int_0^{\infty} r dr \sum_{j=1}^{\infty} r^{2j} h_j(\mathbf{t}) h_j(\mathbf{t}^*) e^{-r^2} \\ &= \pi \int_0^{\infty} dx \sum_{j=1}^{\infty} x^j h_j(\mathbf{t}) h_j(\mathbf{t}^*) e^{-x} \end{aligned} \quad (2.7)$$

and this is just the Borel sum formula for the series (1.14).

**2e. Sketch of a formal proof of Theorem 2.2 using the generating expansion.**

The following identity [Ma] gives a generating function expansion for Schur functions:

$$\exp\left(\sum_{n=0}^{\infty} n t_n t_n^*\right) = \sum_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*), \quad (2.8)$$

Applying this identity to the cases where:

$$\mathbf{t}^* = [\mathbf{x}] := \left(\sum_{i=1}^{\infty} x_i, \frac{1}{2} \sum_{i=1}^{\infty} x_i^2, \dots\right), \quad \mathbf{x} = (x_1, \dots, x_M), \quad (2.9a)$$

$$\mathbf{t} = [\mathbf{y}] := \left(\sum_{i=1}^{\infty} y_i, \frac{1}{2} \sum_{i=1}^{\infty} y_i^2, \dots\right), \quad \mathbf{y} = (y_1, \dots, y_M), \quad (2.9b)$$

gives the expansions

$$\exp\left(\sum_{j=1}^M \sum_{m=0}^{\infty} t_m x_i^m\right) = \sum_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}([\mathbf{x}]), \quad (2.9a)$$

$$\exp\left(\sum_{j=1}^M \sum_{m=0}^{\infty} t_m^* y_j^m\right) = \sum_{\mu} s_{\mu}(\mathbf{t}^*) s_{\mu}([\mathbf{y}]). \quad (2.9b)$$

Substituting these into the expression for the 2-matrix partition function, using the relations

$$s_{\lambda}([\mathbf{x}]) \Delta(\mathbf{x}) = \det(x_i^{n_j - j + M}), \quad s_{\lambda}([\mathbf{y}]) \Delta(\mathbf{y}) = \det(y_k^{m_l - l + M}), \quad (2.11)$$

and once again applying Lemma 2.2, the integrals may be evaluated along suitably chosen contours (as in the formal proof above for  $M = 1$ ), to obtain the stated result:

$$\mathbf{Z}_{2P}^M(\mathbf{t}, \mathbf{t}^*) = \prod_{j=0}^M j! \sum_{\lambda} (M)_{\lambda} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{t}^*) . \quad (2.12)$$

The detailed proof, and a further discussion of the interpretation of the sum (1.12) as an asymptotic series about the Gaussian point may be found in [HO].

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